

Fuzzy stress and strength reliability based on the generalized mixture exponential distribution

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Abstract

In this paper, we will discuss the stress and strength reliability and the fuzzy stress and strength reliability based on the generalized mixtures of exponential distribution. A fuzzy membership function is defined as a function of the difference between stress and strength values and is increasing whenever $\mathbf{X} > \mathbf{Y}$. Several estimation methods, such as Maximize likelihood estimation method, Weighted Least squared estimation method and Percentile estimation method, will be provided to estimate the corresponding measures. Simulation studies are constructed to compare the performances of the proposed estimators. The comparisons are based on the biases, mean squared errors (MSEs) and the efficiency of the estimators. One data analysis has been performed for illustrative purpose.

Keywords: Stress and strength reliability, Fuzzy, Generalized mixtures of exponential distribution, MLE

1 Introduction

Stress and strength reliability analysis often involves the use of probability theory and statistics to quantify the likelihood of failure under different scenarios. For two independent random variables, X and Y , the stress and strength reliability is defined as

$R = P(X > Y)$. If the stress X is greater than strength Y , it may result in component failure or system malfunction. The accurate prediction of stress and strength reliability is crucial for ensuring the safety, performance, and longevity of these systems. Such as, through studying the stress and strength reliability, engineers can design engines that minimize the risk of failure, ensuring the safety of passengers and crew; manufacturers can reduce the risk of component failures, improve vehicle safety, and enhance customer satisfaction. Researchers in this field employ a range of analytical and experimental techniques to study the stress and strength behavior of materials and components. In the context of stress-strength models, Kelley et al. (1976) [1] and Tong (1974, 1977) [2, 3] considered the estimation of stress and strength reliability for two independent exponential random variables. Awad et al. (1981) [4] studied the stress and strength reliability of two bivariate exponential distributions. Kundu and Gupta (2005) [5] dealt with the estimation of $P(X > Y)$ when X and Y are two independent generalized exponential distributions with different shape parameters but having the same scale parameters. Saraçoğlu et al. (2012) [6] studied the estimation of $R = P(X > Y)$ for exponential distribution under progressive type-II censoring. Yazgan et al. (2022) [7] considered the estimation of stress-strength reliability in the presence of fuzziness when the two random variables follow the WE distribution. Kumari et al. (2023) [8] studied the stress-strength reliability for inverse exponentiated distributions.

However, most of the engineering processes inherently have uncertainty that must be dealt with and represented effectively. Sometimes, the data cannot be reported precisely under some unexpected situations. In addition, the subjective evaluation of the lifetime data leads to the fuzziness. In order to overcome these problems, several researchers paid attention to applying the fuzzy set theory, which was introduced by Zadeh (1999) [9]. Such as, Huang (1995) [10] introduced a methodology for the reliability modeling in the presence of fuzziness and extended conventional reliability into fuzzy reliability by using the notion of probability measure of the fuzzy event. Cai (1996) [11] presented an overview on the application of fuzzy methodology by representing the failure in terms of fuzzy sets. Huang et al. (2006) [12] studied on estimates of the parameters and the reliability function of multi-parameter lifetime distributions representing the lifetime data with fuzzy measures. Li and Kapur (2013) [13] proposed fuzzy reliability measures treating success and failure as a fuzzy state. Eryilmaz and Tütüncü (2015) [14] introduced the fuzzy stress-strength reliability for a single unit and defined some properties of fuzzy reliability using a fuzzy membership function which is related to the value of $(x - y)$. Recently, Yazgan et al. (2022) [7] study the reliability of the stress-strength model in the presence of the fuzziness when the stress and strength variables have weighted exponential distribution with the common shape parameter. Hassan and Muse (2023) [15] consider the Bayesian and non-Bayesian estimation of the stress-strength model and the mean remaining strength when there is fuzziness for stress and strength random variables having Lindley's distribution with different parameters. Milošević and Stanojević (2024) [16] introduce a novel membership function for the definition of fuzzy stress-strength reliability of a two component system.

The generalized mixture exponential (GME) distribution, which was proposed by Yang et al. (2021) [17], offered more flexible distributions with applications in lifetime modeling. A random variable X is said to have a GME distribution if its probability density function (pdf) is of the following form,

$$f(x; \lambda, \alpha, \beta) = \frac{(\alpha + 1)\lambda}{\alpha + 1 + \alpha\beta} e^{-\lambda x} [1 + \beta(1 - e^{-\alpha\lambda x})], \quad x > 0, \quad (1)$$

where $\alpha > 0$ is the scale parameter, $\lambda > 0$ and $\beta \geq -1$ are the shape parameters, and we denote it as $X \sim GME(\lambda, \alpha, \beta)$. Moreover, the cumulative distribution function (cdf) of $X \sim GME(\lambda, \alpha, \beta)$ is given by,

$$F(x; \lambda, \alpha, \beta) = 1 + \frac{\beta e^{-(\alpha+1)\lambda x}}{\alpha + 1 + \alpha\beta} - \frac{(\alpha + 1)(\beta + 1)e^{-\lambda x}}{\alpha + 1 + \alpha\beta}.$$

which includes the exponential distribution and weighted exponential distribution. To the best of our knowledge, there has been no previous work handling the reliability and the mean remaining strength models within the framework of fuzziness based on the GME distribution. The rest of the article is organized as follows. The stress and strength reliability based on the GME distribution is introduced in Section 2. Some estimation methods are studied in Section 3. Simulations are conducted to investigate and compare the performances of the proposed estimation methods in Section 4. The real data application is analyzed to illustrate the usefulness of the proposed method in Section 5. Some conclusions are presented in Section 6.

2 Stress and strength reliability

In this section, we will discuss the measure of the stress and strength reliability, R , fuzzy stress and strength reliability and R_F .

Theorem 1. Let $X \sim GME(\lambda_1, \alpha_1, \beta_1)$ and $Y \sim GME(\lambda_2, \alpha_2, \beta_2)$ are independent, and define $R = P(X > Y)$. Then

$$R = \frac{(\alpha_1 + 1)\lambda_1(\alpha_2 + 1)\lambda_2}{(\alpha_1 + 1 + \alpha_1\beta_1)(\alpha_2 + 1 + \alpha_2\beta_2)} \left[\frac{(1 + \beta_1)(1 + \beta_2)}{\lambda_1(\lambda_1 + \lambda_2)} - \frac{\beta_2(1 + \beta_1)}{\lambda_1(\lambda_1 + \lambda_2 + \alpha_2\lambda_2)} \right. \\ \left. - \frac{\beta_1(1 + \beta_2)}{\lambda_1(1 + \alpha_1)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1)} + \frac{\beta_1\beta_2}{\lambda_1(1 + \alpha_1)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1 + \alpha_2\lambda_2)} \right]$$

Proof. Based on the definition of the stress and strength reliability, we can obtain

$$R = P(X > Y) = \iint_{x>y} dF_X(x) dF_Y(y) \\ = \int_0^\infty \int_y^\infty f(x; \lambda_1, \alpha_1, \beta_1) f(y; \lambda_2, \alpha_2, \beta_2) dx dy \\ = \int_0^\infty \int_y^\infty \frac{(\alpha_1 + 1)\lambda_1}{\alpha_1 + 1 + \alpha_1\beta_1} e^{-\lambda_1 x} [1 + \beta_1(1 - e^{-\alpha_1\lambda_1 x})] \frac{(\alpha_2 + 1)\lambda_2}{\alpha_2 + 1 + \alpha_2\beta_2} e^{-\lambda_2 y} [1 + \beta_2(1 - e^{-\alpha_2\lambda_2 y})] dx dy$$

$$\begin{aligned}
&= \frac{(\alpha_1 + 1)\lambda_1(\alpha_2 + 1)\lambda_2}{(\alpha_1 + 1 + \alpha_1\beta_1)(\alpha_2 + 1 + \alpha_2\beta_2)} \int_0^\infty \int_y^\infty e^{-\lambda_1 x} [1 + \beta_1(1 - e^{-\alpha_1 \lambda_1 x})] e^{-\lambda_2 y} [1 + \beta_2(1 - e^{-\alpha_2 \lambda_2 y})] dx dy \\
&= \frac{(\alpha_1 + 1)\lambda_1(\alpha_2 + 1)\lambda_2}{(\alpha_1 + 1 + \alpha_1\beta_1)(\alpha_2 + 1 + \alpha_2\beta_2)} \left[\frac{(1 + \beta_1)(1 + \beta_2)}{\lambda_1(\lambda_1 + \lambda_2)} - \frac{\beta_2(1 + \beta_1)}{\lambda_1(\lambda_1 + \lambda_2 + \alpha_2\lambda_2)} \right. \\
&\quad \left. - \frac{\beta_1(1 + \beta_2)}{\lambda_1(1 + \alpha_1)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1)} + \frac{\beta_1\beta_2}{\lambda_1(1 + \alpha_1)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1 + \alpha_2\lambda_2)} \right]
\end{aligned}$$

□

Corollary 2.1. *Let $X \sim GME(\lambda_1, \alpha_1, \beta_1)$ and $Y \sim GME(\lambda_2, \alpha_2, \beta_2)$ are independent, we have,*

(i) *For $\beta_1 = \beta_2 = -1$,*

$$R = \frac{\lambda_2(1 + \alpha_2)}{\lambda_1 + \lambda_2 + \alpha_1\lambda_1 + \alpha_2\lambda_2}.$$

(ii) *For $\beta_1 \rightarrow \infty$ and $\beta_2 \rightarrow \infty$,*

$$\begin{aligned}
R &= \frac{(\alpha_1 + 1)(\alpha_2 + 1)\lambda_2^2}{\alpha_1} \left[\frac{1}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + \alpha_2\lambda_2)} \right. \\
&\quad \left. - \frac{1}{(1 + \alpha_1)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1 + \alpha_2\lambda_2)} \right].
\end{aligned}$$

(iii) *For $\lambda_1 = \lambda_2$,*

$$\begin{aligned}
R &= \frac{(\alpha_1 + 1)(\alpha_2 + 1)}{(\alpha_1 + 1 + \alpha_1\beta_1)(\alpha_2 + 1 + \alpha_2\beta_2)} \left[\frac{(1 + \beta_1)(1 + \beta_2)}{2} \right. \\
&\quad \left. - \frac{\beta_2(1 + \beta_1)}{2 + \alpha_2} - \frac{\beta_1(1 + \beta_2)}{2 + \alpha_2} + \frac{\beta_1\beta_2}{(1 + \alpha_1)(2 + \alpha_1\alpha_2)} \right].
\end{aligned}$$

(iv) *For $\lambda_1 = \lambda_2$, $\alpha_1 = \alpha_2$, and $\beta_1 = \beta_2$, we have $R = \frac{1}{2}$.*

According to Yang et al.(2021) [17], the $GME(\lambda, \alpha, \beta)$ distribution is reduced to $WE(\lambda, \alpha)$ distribution. Moreover, if set $\alpha_1 = \alpha_2 = \alpha$, we can obtain

$$R = \frac{\lambda_2^2 [(3 + 3\alpha + \alpha^2)\lambda_1 + (1 + \alpha)\lambda_2]}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + \alpha\lambda_1)(\lambda_1 + \lambda_2 + \alpha\lambda_2)},$$

which is the same as equation (7) in [7].

For different values of λ_1 , β_1 , λ_2 , β_2 , the values of R are plotted in the Figure 1, which indicate that the R can generate curves with various shapes. According to the figure 1, we can see that the values of R increase as the values of β increase, however, the trend is opposite for the α .

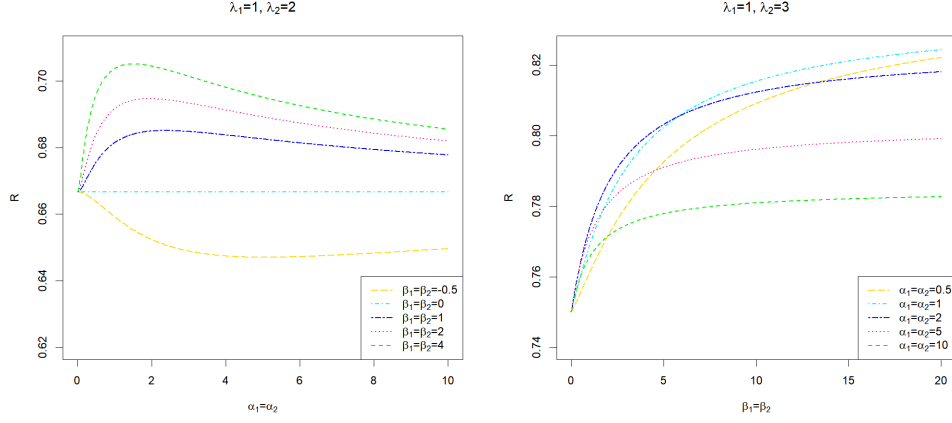


Fig. 1 The R curves for different parameters.

In the following, we discuss the fuzzy stress and strength reliability of two independent GME distribution, which is defined as follows,

$$R_F = P(X \succ Y) = \iint_{x>y} \mu_{A(y)}(x) dF_X(x) dY(y), \quad (2)$$

where $A(y) = \{x : x > y\}$, and

$$\mu_{A(y)}(x) = \begin{cases} 0, & y \geq x \\ 1 - e^{-k(x-y)}, & y < x, \end{cases}$$

where $k \in \mathbb{R}$.

Theorem 2. Let $X \sim GME(\lambda_1, \alpha_1, \beta_1)$ and $Y \sim GME(\lambda_2, \alpha_2, \beta_2)$ are independent. Then the fuzzy stress and strength reliability is

$$R_F = \frac{k(\alpha_1 + 1)\lambda_1(\alpha_2 + 1)\lambda_2}{(\alpha_1 + 1 + \alpha_1\beta_1)(\alpha_2 + 1 + \alpha_2\beta_2)} \left[\frac{(1 + \beta_1)(1 + \beta_2)}{\lambda_1(\lambda_1 + k)(\lambda_1 + \lambda_2)} - \frac{\beta_2(1 + \beta_1)}{\lambda_1(\lambda_1 + k)(\lambda_1 + \lambda_2 + \alpha_2\lambda_2)} \right. \\ \left. - \frac{\beta_1(1 + \beta_2)}{\lambda_1(1 + \alpha_1)(\lambda_1 + \alpha_1\lambda_1 + k)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1)} \right. \\ \left. + \frac{\beta_1\beta_2}{\lambda_1(1 + \alpha_1)(\lambda_1 + \alpha_1\lambda_1 + k)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1 + \alpha_2\lambda_2)} \right].$$

Proof. Based on the Equation 2, we can get that

$$R_F = \frac{(\alpha_1 + 1)\lambda_1(\alpha_2 + 1)\lambda_2}{(\alpha_1 + 1 + \alpha_1\beta_1)(\alpha_2 + 1 + \alpha_2\beta_2)} \int_0^\infty \int_y^\infty (1 - e^{-k(x-y)}) e^{-\lambda_1 x} [1 + \beta_1(1 - e^{-\alpha_1\lambda_1 x})] e^{-\lambda_2 y} [1 + \beta_2(1 - e^{-\alpha_2\lambda_2 y})] dx dy$$

$$\begin{aligned}
&= \frac{k(\alpha_1 + 1)\lambda_1(\alpha_2 + 1)\lambda_2}{(\alpha_1 + 1 + \alpha_1\beta_1)(\alpha_2 + 1 + \alpha_2\beta_2)} \left[\frac{(1 + \beta_1)(1 + \beta_2)}{\lambda_1(\lambda_1 + k)(\lambda_1 + \lambda_2)} - \frac{\beta_2(1 + \beta_1)}{\lambda_1(\lambda_1 + k)(\lambda_1 + \lambda_2 + \alpha_2\lambda_2)} \right. \\
&\quad \left. - \frac{\beta_1(1 + \beta_2)}{\lambda_1(1 + \alpha_1)(\lambda_1 + \alpha_1\lambda_1 + k)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1)} \right. \\
&\quad \left. + \frac{\beta_1\beta_2}{\lambda_1(1 + \alpha_1)(\lambda_1 + \alpha_1\lambda_1 + k)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1 + \alpha_2\lambda_2)} \right].
\end{aligned}$$

□

For different values of k , the values of $R_{F,k}$ versus λ , α , and β are presented in the Figure 2, which indicate that the $R_{F,k}$ can generate curves with various shapes.

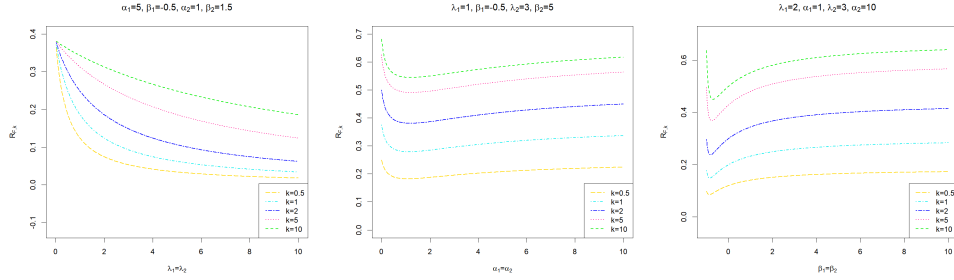


Fig. 2 The $R_{F,k}$ curves for different parameters and k .

Corollary 2.2. Let $X \sim GME(\lambda_1, \alpha_1, \beta_1)$ and $Y \sim GME(\lambda_2, \alpha_2, \beta_2)$ are independent, we can have

(i) For $\beta_1 = \beta_2 = -1$,

$$R_F = \frac{k\lambda_2(1 + \alpha_2)}{(\lambda_1 + \alpha_1\lambda_1 + k)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1 + \alpha_2\lambda_2)}$$

(ii) For $\beta_1 \rightarrow \infty, \beta_2 \rightarrow \infty$

$$\begin{aligned}
R_F &= \frac{k(\alpha_1 + 1)(\alpha_2 + 1)\lambda_2^2}{\alpha_1} \left[\frac{1}{(\lambda_1 + k)(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + \alpha_2\lambda_2)} \right. \\
&\quad \left. - \frac{1}{(1 + \alpha_1)(\lambda_1 + \alpha_1\lambda_1 + k)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1)(\lambda_1 + \lambda_2 + \alpha_1\lambda_1 + \alpha_2\lambda_2)} \right]
\end{aligned}$$

According to Yang et al.(2021) [17], the $GME(\lambda, \alpha, \beta)$ distribution is reduced to $E(\lambda(\alpha + 1))$ distribution as $\beta = -1$, which the same result as Eryilmaz and Tütüncü (2015) [14]. Besides, for different k , we denote $R_{F,k}$ to its corresponding R_F value for easy representation, and as $k \rightarrow \infty$, $R_{F,k}$ approaches to R_F .

3 Methods of Estimation

In this section, we consider the methods of maximum likelihood, percentile, and weighted least-square to estimate the unknown parameters, $\theta = (\lambda, \alpha, \beta)$, of the GME distribution. Suppose $x_1, x_2, \dots, x_n$ is a random sample from $GME(\lambda, \alpha, \beta)$.

3.1 Maximum Likelihood Estimator

The method of maximum likelihood is the most frequently used method for parameter estimation. According to the Equation (1), the likelihood function is calculated as

$$L(\lambda, \alpha, \beta | x_1, \dots, x_n) = \left(\frac{(\alpha + 1)\lambda}{\alpha + 1 + \alpha\beta} \right)^n e^{-\lambda \sum_{i=1}^n x_i} \prod_{i=1}^n [1 + \beta(1 - e^{-\alpha\lambda x_i})].$$

The log-likelihood function is given by

$$\begin{aligned} \ell(\lambda, \alpha, \beta | x_1, \dots, x_n) &= n[\log(\alpha + 1) + \log(\lambda) - \log(\alpha + 1 + \alpha\beta)] - \lambda \sum_{i=1}^n x_i \\ &\quad + \sum_{i=1}^n \log[1 + \beta(1 - e^{-\alpha\lambda x_i})]. \end{aligned} \quad (3)$$

We denote the first partial derivatives of (3) by ℓ_λ , ℓ_α and ℓ_β . Setting $\ell_\lambda = 0$, $\ell_\alpha = 0$, and $\ell_\beta = 0$, we have

$$\begin{aligned} \ell_\lambda &= \frac{n}{\lambda} - \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{\beta\alpha x_i e^{-\alpha\lambda x_i}}{1 + \beta(1 - e^{-\alpha\lambda x_i})} = 0, \\ \ell_\alpha &= \frac{n}{\alpha + 1} - \frac{n(1 + \beta)}{\alpha + 1 + \alpha\beta} + \sum_{i=1}^n \frac{\beta\lambda x_i e^{-\alpha\lambda x_i}}{1 + \beta(1 - e^{-\alpha\lambda x_i})} = 0, \\ \ell_\beta &= -\frac{n\alpha}{\alpha + 1 + \alpha\beta} + \sum_{i=1}^n \frac{1 - e^{-\alpha\lambda x_i}}{1 + \beta(1 - e^{-\alpha\lambda x_i})} = 0. \end{aligned}$$

The maximum likelihood estimator (MLE) $\hat{\theta}$ of the unknown parameters θ can be obtained by optimizing the log-likelihood function with respect to the involved parameters. Due to the non-linearity of these equations, the MLEs of parameters can be obtained numerically. These estimators can be easily obtained by using the functions from the statistical software R.

3.2 Percentile Estimator

The percentile estimators (PE) are mainly obtained by minimizing the distance between the sample percentile and population percentile points. In the following, we apply this approach on the GME distribution to obtain the parameter estimators. Suppose $F(x_{(j)})$ denotes the distribution function of the ordered random variables

$x_{(1)} < \dots < x_{(n)}$. Let $P_i = \frac{i}{n+1}$ be the unbiased estimator of $F(x_{(i)}; \lambda, \alpha, \beta)$. Define the following function

$$h(\lambda, \alpha, \beta) = \sum_{i=1}^n [\ln P_i - \ln(F(x_{(i)}; \lambda, \alpha, \beta))]^2, \quad (4)$$

where $F(x; \lambda, \alpha, \beta) = \frac{\beta}{\alpha + 1 + \alpha\beta} [e^{-\lambda(1+\alpha)x} - e^{-\lambda x}] - e^{-\lambda x} + 1$. The PE of θ can be obtained by minimizing $h(\lambda, \alpha, \beta)$. Therefore, $\hat{\theta}$ can be obtained by solving the following equations,

$$\begin{aligned} \frac{\partial h(\lambda, \alpha, \beta)}{\partial \lambda} &= \sum_{i=1}^n \left\{ 2Q_i \left\{ \frac{\beta}{\alpha + 1 + \alpha\beta} [-(1+\alpha)C_i + B_i] + B_i \right\} \right\} = 0, \\ \frac{\partial h(\lambda, \alpha, \beta)}{\partial \alpha} &= \sum_{i=1}^n \left\{ 2Q_i \left\{ \frac{-(1+\beta)}{(\alpha + 1 + \alpha\beta)^2} [e^{-\lambda(1+\alpha)X_{(i)}} - e^{-\lambda X_{(i)}}] - \frac{\beta\lambda}{\alpha + 1 + \alpha\beta} C_i \right\} \right\} = 0, \\ \frac{\partial h(\lambda, \alpha, \beta)}{\partial \beta} &= \sum_{i=1}^n \left\{ 2Q_i \left\{ \frac{1-\alpha}{(\alpha + 1 + \alpha\beta)^2} [e^{-\lambda(1+\alpha)X_{(i)}} - e^{-\lambda X_{(i)}}] \right\} \right\} = 0, \end{aligned}$$

where $Q_i = \frac{\beta}{\alpha + 1 + \alpha\beta} [e^{-\lambda(1+\alpha)x_{(i)}} - e^{-\lambda x_{(i)}}] - e^{-\lambda x_{(i)}} + 1 - \frac{i}{n+1}$, $B_i = x_{(i)} e^{-\lambda x_{(i)}}$ and $C_i = x_{(i)} e^{-\lambda(1+\alpha)x_{(i)}}$.

3.3 Weighted Least-Square Estimator

The weighted least-square estimator (WLS) is an extension of LS and proposed by Swain et al. (1988)[18], which studied the WLS is obtained by minimizing the function

$$W(\lambda, \alpha, \beta) = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{(i)}; \lambda, \alpha, \beta) - \frac{i}{n+1} \right]^2,$$

where $F(\cdot)$ function has been given in Equation (4). Therefore, the WLS estimator of θ can be obtained by

$$\begin{aligned} \frac{\partial W(\lambda, \alpha, \beta)}{\partial \lambda} &= \sum_{i=1}^n \left\{ \frac{2(n-1)^2(n+2)}{i(n-i+1)} Q_i \left\{ \frac{\beta}{\alpha + 1 + \alpha\beta} [-(1+\alpha)C_i + B_i] + B_i \right\} \right\} = 0, \\ \frac{\partial W(\lambda, \alpha, \beta)}{\partial \alpha} &= \sum_{i=1}^n \left\{ \frac{2(n-1)^2(n+2)}{i(n-i+1)} Q_i \left\{ \frac{-(1+\beta)}{(\alpha + 1 + \alpha\beta)^2} [e^{-\lambda(1+\alpha)x_{(i)}} - e^{-\lambda x_{(i)}}] \right. \right. \\ &\quad \left. \left. - \frac{\beta\lambda}{\alpha + 1 + \alpha\beta} C_i \right\} \right\} = 0, \\ \frac{\partial W(\lambda, \alpha, \beta)}{\partial \beta} &= \sum_{i=1}^n \left\{ \frac{2(n-1)^2(n+2)}{i(n-i+1)} Q_i \left\{ \frac{1-\alpha}{(\alpha + 1 + \alpha\beta)^2} [e^{-\lambda(1+\alpha)x_{(i)}} - e^{-\lambda x_{(i)}}] \right\} \right\} = 0, \end{aligned}$$

where Q_i, B_i and $C_i, i = 1, \dots, n$, are defined as above.

4 Simulation studies

In this section, we would like to study the stress and strength reliability and fuzzy reliability of the GME distribution based on different estimation methods. All the programs are conducted by the R programming. Simulation studies are designed with $N = 1000$ repetitions for different values of $\theta = (\lambda, \alpha, \beta)$. To address the performance of the proposed method, the bias and mean squared error(MSE) are obtained by the following

$$\begin{aligned} Bias(\hat{\theta}) &= \frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta), \\ MSE(\hat{\theta}) &= \frac{1}{N} \sum_{i=1}^N (\hat{\theta}_i - \theta)^2, \\ SD(\hat{\theta}) &= \sqrt{\frac{1}{N-1} \sum_{i=1}^N (\hat{\theta}_i - \theta)^2}. \end{aligned}$$

We take the sample size $n = \{20, 50, 100, 200, 300, 400, 500\}$ for each simulation, and consider $k = \{1, 10, 100\}$ for R_F as Yazgan et al. (2022) [7]. Furthermore, we consider three scenarios as: (i) $\theta_X = (1, 5, -0.5)$ and $\theta_Y = (2, 1, 1.5)$; (ii) $\theta_X = (1, 5, -0.5)$ and $\theta_Y = (3, 10, 5)$; (iii) $\theta_X = (2, 1, 1.5)$ and $\theta_Y = (3, 10, 5)$, where θ_X and θ_Y are the parameters of θ for data X and Y , respectively. The simulation results are shown in Tables 1-3.

Table 1 The mean value, sd, bias, and MSE for scenario 1 of estimators under different n .

Sample Size	Method	R									
		$k = 1$					$k = 10$				
		Mean	SD	Bias	MSE		Mean	SD	Bias	MSE	
$n = 20$	MLE	0.512	0.094	-0.037	0.100	0.247	0.072	-0.018	0.074	0.454	0.095
	PE	0.456	0.151	-0.093	0.177	0.198	0.104	-0.067	0.123	0.394	0.151
	WLS	0.516	0.079	-0.033	0.085	0.260	0.061	-0.005	0.061	0.460	0.079
$n = 50$	MLE	0.540	0.048	-0.009	0.048	0.257	0.037	-0.008	0.038	0.480	0.048
	Per	0.504	0.074	-0.045	0.086	0.231	0.058	-0.034	0.067	0.444	0.074
	WLS	0.531	0.048	-0.018	0.051	0.260	0.037	-0.005	0.037	0.473	0.048
$n = 100$	MLE	0.546	0.037	-0.003	0.037	0.266	0.029	0.001	0.029	0.489	0.037
	PE	0.521	0.049	-0.028	0.060	0.246	0.041	-0.019	0.045	0.462	0.051
	WLS	0.539	0.039	-0.010	0.040	0.263	0.030	-0.002	0.030	0.481	0.039
$n = 200$	MLE	0.547	0.027	-0.002	0.027	0.263	0.019	-0.002	0.019	0.489	0.027
	Per	0.530	0.034	-0.019	0.039	0.254	0.028	-0.011	0.030	0.472	0.035
	WLS	0.540	0.026	-0.009	0.028	0.261	0.019	-0.004	0.020	0.482	0.026
$n = 300$	MLE	0.548	0.022	-0.001	0.022	0.264	0.017	-0.001	0.017	0.450	0.023
	PE	0.536	0.029	-0.013	0.031	0.258	0.024	-0.007	0.025	0.478	0.029
	WLS	0.543	0.023	-0.006	0.024	0.263	0.017	-0.002	0.017	0.485	0.023
$n = 400$	MLE	0.547	0.020	-0.002	0.020	0.263	0.014	-0.002	0.014	0.489	0.020
	Per	0.538	0.023	-0.011	0.026	0.259	0.021	-0.006	0.022	0.480	0.024
	WLS	0.545	0.020	-0.004	0.021	0.263	0.015	-0.002	0.015	0.487	0.020
$n = 500$	MLE	0.548	0.018	-0.001	0.018	0.264	0.013	-0.001	0.013	0.490	0.018
	PE	0.540	0.024	-0.009	0.025	0.261	0.024	-0.004	0.024	0.482	0.024
	WLS	0.547	0.018	-0.002	0.018	0.265	0.013	0.000	0.013	0.489	0.017

Table 2 The mean value, SD, Bias, and MSE for scenario 2 of estimators under different n .

Sample Size	Method	R									
		$k = 1$					$k = 10$				
		Mean	SD	Bias	MSE		Mean	SD	Bias	MSE	
$n = 20$	MLE	0.640	0.085	-0.028	0.089	0.297	0.070	0.090	-0.033	0.094	0.631
	PE	0.631	0.154	-0.121	0.193	0.214	0.101	0.154	-0.136	0.203	0.537
	WLS	0.645	0.066	-0.024	0.069	0.312	0.051	0.066	-0.029	0.071	0.635
$n = 50$	MLE	0.648	0.047	-0.021	0.051	0.311	0.040	0.049	-0.020	0.053	0.640
	PE	0.643	0.057	-0.026	0.062	0.289	0.056	0.062	-0.034	0.070	0.633
	WLS	0.643	0.050	-0.026	0.055	0.305	0.040	0.051	-0.029	0.058	0.634
$n = 100$	MLE	0.662	0.039	-0.007	0.039	0.317	0.031	0.041	-0.007	0.042	0.654
	PE	0.638	0.038	-0.031	0.049	0.287	0.040	0.042	-0.037	0.056	0.629
	WLS	0.663	0.031	-0.006	0.031	0.321	0.026	0.032	-0.007	0.032	0.654
$n = 200$	MLE	0.666	0.027	-0.003	0.027	0.317	0.024	0.029	-0.004	0.029	0.657
	Per	0.660	0.032	-0.009	0.033	0.313	0.027	0.033	-0.011	0.035	0.651
	WLS	0.663	0.020	-0.006	0.021	0.318	0.018	0.021	-0.008	0.022	0.654
$n = 300$	MLE	0.667	0.021	-0.002	0.021	0.318	0.017	0.022	-0.002	0.022	0.659
	PE	0.659	0.024	-0.010	0.026	0.312	0.023	0.0255	-0.012	0.028	0.650
	WLS	0.664	0.018	-0.005	0.019	0.317	0.017	0.020	-0.008	0.021	0.655
$n = 400$	MLE	0.667	0.018	-0.002	0.019	0.317	0.015	0.019	-0.003	0.020	0.658
	Per	0.663	0.023	-0.006	0.024	0.316	0.020	0.024	-0.007	0.025	0.654
	WLS	0.667	0.019	-0.002	0.019	0.318	0.017	0.020	-0.004	0.020	0.658
$n = 500$	MLE	0.667	0.017	-0.002	0.017	0.317	0.013	0.018	-0.003	0.018	0.658
	PE	0.664	0.023	-0.005	0.024	0.317	0.020	0.024	-0.006	0.025	0.655
	WLS	0.663	0.016	-0.006	0.017	0.314	0.011	0.016	-0.008	0.017	0.654

Table 3 The mean value, sd, bias, and MSE for scenario 3 of estimators under different n .

Sample Size	Method	R					R _F														
		-					k = 1					k = 10					k = 100				
		Mean	SD	Bias	MSE		Mean	sd	Bias	MSE		Mean	SD	Bias	MSE		Mean	SD	Bias	MSE	
n = 20	MLE	0.626	0.067	-0.024	0.069		0.220	0.042	-0.019	0.044		0.532	0.065	-0.029	0.069		0.615	0.067	-0.025	0.069	
	PE	0.667	0.027	0.018	0.030		0.240	0.032	0.001	0.029		0.571	0.032	0.009	0.030		0.657	0.027	0.017	0.029	
	WLS	0.655	0.093	0.005	0.088		0.247	0.066	0.008	0.062		0.566	0.094	0.005	0.089		0.645	0.093	0.005	0.088	
n = 50	MLE	0.642	0.056	-0.008	0.055		0.239	0.045	-0.000	0.044		0.553	0.060	-0.008	0.059		0.632	0.056	-0.008	0.056	
	PE	0.635	0.073	-0.015	0.072		0.239	0.045	0.000	0.043		0.550	0.072	-0.012	0.071		0.626	0.073	-0.014	0.072	
	WLS	0.657	0.041	0.007	0.041		0.248	0.025	0.008	0.026		0.569	0.040	0.007	0.026		0.569	0.041	0.007	0.041	
n = 100	MLE	0.646	0.033	-0.003	0.032		0.240	0.026	0.001	0.026		0.558	0.035	-0.003	0.034		0.637	0.033	-0.003	0.033	
	PE	0.645	0.051	-0.004	0.051		0.240	0.037	0.001	0.036		0.556	0.053	-0.005	0.052		0.635	0.052	-0.005	0.051	
	WLS	0.658	0.041	0.009	0.042		0.250	0.030	0.011	0.032		0.571	0.043	0.009	0.044		0.649	0.042	0.009	0.042	
n = 200	MLE	0.645	0.027	-0.004	0.027		0.238	0.018	-0.001	0.018		0.557	0.027	-0.004	0.027		0.636	0.027	-0.004	0.027	
	PE	0.651	0.034	0.001	0.034		0.246	0.024	0.007	0.024		0.564	0.035	0.003	0.034		0.641	0.034	0.001	0.034	
	WLS	0.651	0.022	0.002	0.022		0.242	0.016	0.003	0.017		0.563	0.023	0.002	0.023		0.642	0.022	0.002	0.022	
n = 300	MLE	0.648	0.022	-0.002	0.022		0.238	0.016	-0.001	0.015		0.559	0.023	-0.002	0.023		0.659	0.022	-0.002	0.022	
	PE	0.659	0.065	0.009	0.065		0.251	0.038	0.012	0.040		0.572	0.065	0.011	0.066		0.650	0.065	0.010	0.066	
	WLS	0.650	0.021	0.001	0.021		0.241	0.015	0.002	0.015		0.562	0.022	0.000	0.022		0.641	0.021	0.001	0.021	
n = 400	MLE	0.649	0.018	-0.001	0.018		0.239	0.012	0.000	0.012		0.560	0.018	-0.001	0.018		0.649	0.018	-0.001	0.018	
	PE	0.653	0.020	0.004	0.020		0.244	0.014	0.006	0.015		0.565	0.020	0.004	0.020		0.644	0.020	0.004	0.020	
	WLS	0.650	0.019	0.001	0.019		0.241	0.013	0.002	0.013		0.562	0.019	0.001	0.019		0.641	0.019	0.001	0.019	
n = 500	MLE	0.650	0.017	0.001	0.017		0.240	0.011	0.001	0.011		0.562	0.017	0.001	0.017		0.641	0.017	0.001	0.017	
	PE	0.654	0.020	0.004	0.020		0.247	0.014	0.008	0.016		0.567	0.020	0.005	0.021		0.644	0.020	0.004	0.020	
	WLS	0.652	0.017	0.002	0.017		0.241	0.011	0.002	0.011		0.563	0.017	0.002	0.017		0.644	0.017	0.002	0.017	

From Tables 1-3, we find that the values of SDs of all three estimators decrease with the increase of sample size n , and the estimated values obtained from these three estimators are close to the true value. This means that when n is large, all estimators will tend to have higher accuracy. For the selected evaluation metrics Bias and MSE, we can see that as n increases, they both approach 0, indicating that these estimators are asymptotically unbiased and consistent with the parameters.

In addition, in order to observe the changes more intuitively, we plotted the values of Bias and MSE for the estimators with different sample sizes n , as shown in Figures 3-4. Furthermore, by comparing the three estimation methods, we can find that the MLE method performs the best, followed by the PE method and the WLS Method.

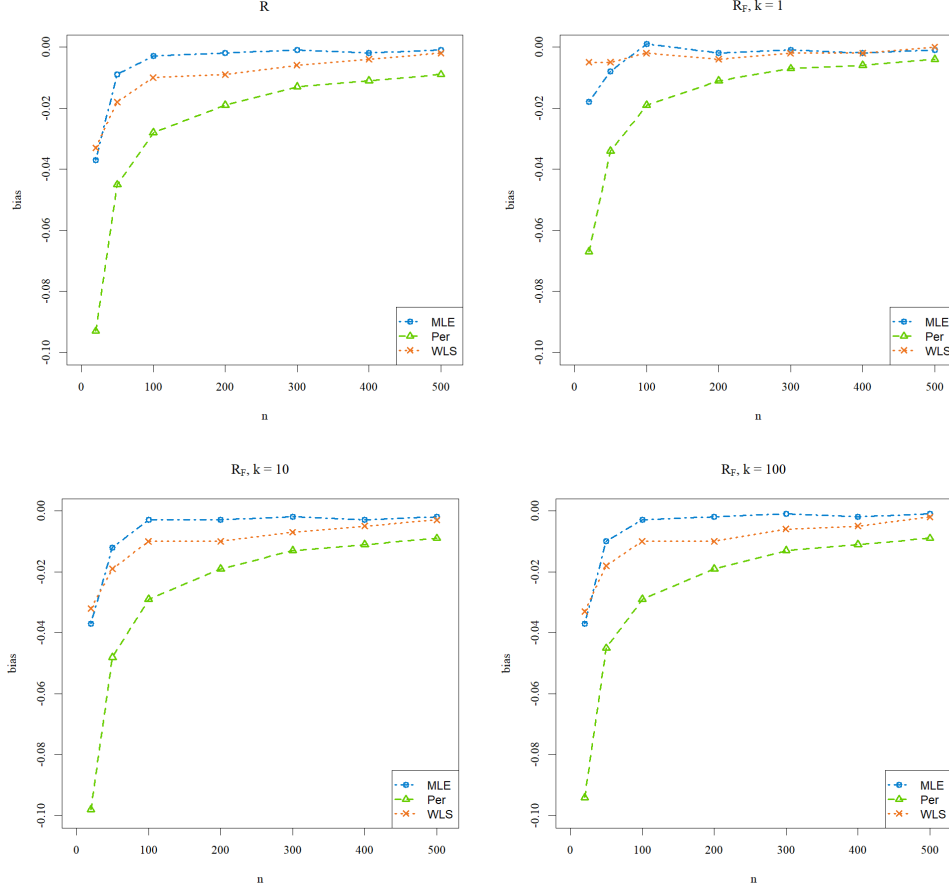


Fig. 3 Bias of estimators versus sample size n for scenario 1.

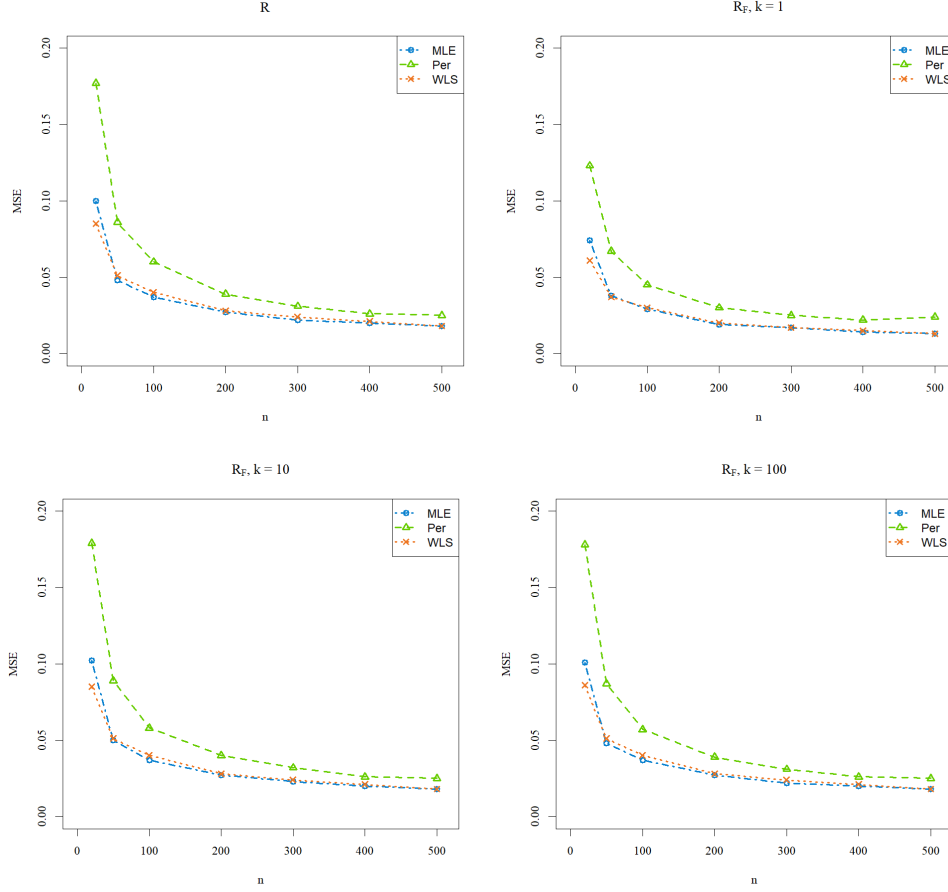


Fig. 4 MSE of estimators versus sample size n for scenario 1.

5 Application

In this section, we consider a real data set which is about the breaking strength of jute fibre (BSJF) provided by Xia et al. (2009) [19] for illustrative purposes. Each fiber group is made up of 30 fibers. Let X be the breaking strength of jute fibre at 10mm, and Y is the breaking strength of jute fibre at 20mm. The detailed information of the data is displayed in Table 4.

To check the validity of the GME distribution with data for the 10mm and 20mm, we apply Anderson-Darling goodness of fit test when the parameters are obtained by the maximum likelihood method. The values of Anderson-Darling statistic (AD), adjusted AD statistic (AD*) and the corresponding p-value for the GME distribution are presented in Table 5.

From Table 5, it can be seen that the p-values of X and Y are both higher than 0.05, which cannot reject the null-hypothesis of distribution. This indicates that the

Table 4 The BSJF data set.

No.	10mm(X)	20mm(Y)	No.	10mm(X)	20mm(Y)	No.	10mm(X)	20mm(Y)
1	693.73	71.46	11	671.49	578.62	21	262.90	547.44
2	704.66	419.02	12	183.16	756.70	22	353.24	116.99
3	323.83	284.64	13	257.44	594.29	23	422.11	375.81
4	778.17	585.57	14	727.23	166.49	24	43.93	581.60
5	123.06	456.60	15	291.27	99.72	25	590.48	119.86
6	637.66	113.85	16	101.15	707.36	26	212.13	48.01
7	383.43	187.85	17	376.42	765.14	27	303.90	200.16
8	151.48	688.16	18	163.40	187.13	28	506.60	36.75
9	108.94	662.66	19	141.38	145.96	29	530.55	244.53
10	50.16	45.58	20	700.74	350.70	30	177.25	83.55

Table 5 The values of AD, AD* and the corresponding p-value of the BSJF data.

	X	Y
AD value	0.692	0.572
AD* value	0.711	0.587
p-value	0.064	0.126

BSJF data set follow the GME distribution. Therefore, we used the MLE method, the PE method, and the WLS method to calculate the R and R_F at different values of k , and the results are shown in Table 6. It can be seen from the table that the estimated values obtained by the three methods are not significantly different, indicating that the three methods perform well.

Table 6 The R , $R_{F,1}$, $R_{F,10}$ and $R_{F,100}$ of different estimation methods for the BSJF data.

Method	R	$R_{F,1}$	$R_{F,10}$	$R_{F,100}$
MLE	0.514	0.512	0.513	0.514
Per	0.538	0.537	0.538	0.538
WLS	0.525	0.524	0.525	0.525

In Table 6, we can find that the estimated values of R are greater than the values of $R_{F,k}$ for all the estimation methods. Also, as the values of k increase, the fuzzy estimators of $R_{F,k}$ are approximately equal to the classical estimators R .

6 Conclusion

This article considers the estimation of stress and strength reliability and fuzzy stress and strength reliability based on a new life time data model. The values of R and R_F are obtained for the generalized mixture exponential distribution with different

values of parameters. Three different estimation methods are studied and the simulation studies show that the MLE method has the best performance than the others. Finally, the real data application shows that the reliability and fuzzy reliability are almost identical. In the future, the reliability R_F and $R_{F,k}$ based on the multivariate distribution would have great significance, and we will continue to study it.

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Declarations

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The authors declare that they have no competing interests.

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